Lecture 8: Cryptography

Trust No One.

Cryptography: Basic Set Up

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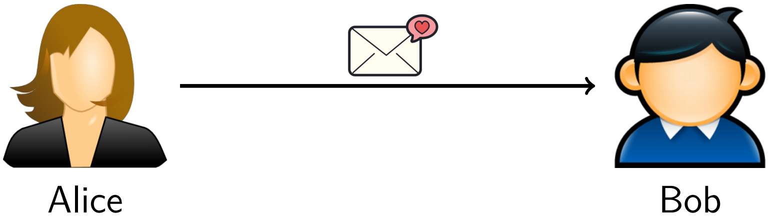
Alice

Cryptography: Basic Set Up

Alice

Bob

Cryptography: Basic Set Up



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Goal: system st Bob gets the message, Eve doesn't

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$b = 0$ doesn't flip, $b = 1$ flips twice

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Does Bob receive the message correctly?
Can Eve read the message?

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w/o pad, c says nothing about m!

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Idea: don't assume shared secret key
Have separate private (only Bob) and public keys

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- B publicizes $N = pq$ and e
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Theorem: Let p be a prime and $a \not\equiv 0 \pmod{p}$. Then $a^{p-1} \equiv 1 \pmod{p}$.

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- Multiply by $\prod_i i^{-1}$, get $1 \equiv a^{p-1} \pmod{p}$

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- Inverse is $f^{-1}(y) = a^{-1}y \pmod{p}$
  - $f^{-1}(f(x)) \equiv a^{-1}ax \equiv x \pmod{p}$
  - $f(f^{-1}(x)) \equiv aa^{-1}x \equiv x \pmod{p}$

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- $m^{ed} \equiv m \pmod{pq}$ is solution to those two
- CRT: m is *only* solution!

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Decryption: need to find $c^d \pmod{N}$

- Again use repeated squaring!

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Note: can quickly test primality

Time For A Break

4 minute breather!

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Today's Discussion Question:
What is the best kind of sandwich?

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Note: can factor quickly on quantum computers
Not an immediate issue, but may be in the future!

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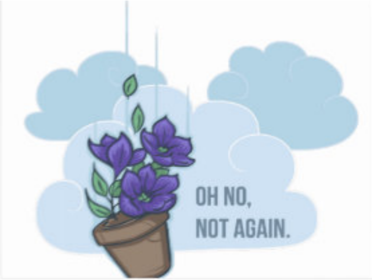

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Moral: even secure protocol can be vulnerable!

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Only Amazon can sign consistently!
Ability to sign $\equiv$ ability to decrypt

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Moral: don't sign arbitrary messages

Fin

Next time: polynomials!