Lecture 9: Polynomials

Why Only Have One Nomial?

What Is a Polynomial?

High school: $p(x) = c_d x^d + c_{d-1} x^{d-1} + \ldots + c_1 x + c_0$

- $d \in \mathbb{N}$ is the *degree*
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Today, prove that these are equivalent!

Polynomial Long Division

Theorem: Let $p(x)$, $d(x)$ be polys. Then $\exists$ $q(x)$, $r(x)$ st $p(x) = q(x)d(x) + r(x)$ and $\deg(r) < \deg(p)$.

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x^2 - 1 & \overline{\big)\, x^4 + 3x^3 - 2x^2 + 0x + 4} \\
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\begin{array}{r}
x^2 + 3x \phantom{{}+0x+4} \\
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- Thus $p(x) = (x - a)q(x)$

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- Else can factor as $(x - a)q(x)$
- 1 root from $(x - a)$, $\leq k$ from $q(x)$
- Total $\leq k + 1$ roots

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But do any $d + 1$ points work?

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Choose c st $p(4) = -\frac{2}{3} \cdot 4 + c = 2$
So $c = 2 + \frac{8}{3} = \frac{14}{3}$

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So $-\frac{2}{3}x + \frac{14}{3}$ is unique degree 1 poly!

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How do we find the Δ_is?

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For 2, take $q_2(x) = (x-0)(x+1) = x^2 + x$
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For -1, take $q_{-1} = (x-0)(x-2) = x^2 - 2x$
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- To construct $\Delta_i(x)$:
  - Take $q_i(x) = \prod_{j \neq i}(x - x_j)$
  - Let $\Delta_i(x) = \frac{q_i(x)}{q_i(x_i)}$

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Note similarities to CRT!

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$$\Delta_1(x) = \frac{(x-6)(x-7)}{(1-6)(1-7)} = \frac{1}{30}x^2 - \frac{13}{30}x + \frac{42}{30}$$

$$\Delta_6(x) = \frac{(x-1)(x-7)}{(6-1)(6-7)} = -\frac{1}{5}x^2 + \frac{8}{5}x - \frac{7}{5}$$

$$\Delta_7(x) = \frac{(x-1)(x-6)}{(7-1)(7-6)} = \frac{1}{6}x^2 - \frac{7}{6}x + 1$$

So take $p(x) = 6\Delta_1(x) + 1\Delta_6(x) + 0\Delta_7(x)$

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Notice: doesn't have to be degree *exactly* 2!

Break

Break time! Talk to your neighbors!

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Today's Discussion Question:
What is your favorite breakfast food?

Get Real

So far, working with polynomials in $\mathbb{R}$

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Numbers modulo a prime is a field!

Finite Fields

Numbers mod p often denoted $GF(p)$[1]

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- Go through $(0,0)$ means $c_0 = 0$
- Go through $(3,1)$ means $c_1 \cdot 3 \equiv 1 \pmod 6$

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Want deg 2 poly mod 7 through $(0, 3)$, $(2, 2)$, $(3, 0)$

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$q_0(x) = (x-2)(x-3) = x^2 - 5x + 6 \equiv x^2 + 2x + 6$
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Don't have to calculate $\Delta_3(x)$ — multiplied by zero!

Take $p(x) = 3\Delta_0(x) + 2\Delta_2(x) + 0\Delta_3(x)$
$\equiv (4x^2 + x + 3) + (6x^2 + 3x) \equiv 3x^2 + 4x + 3 \pmod 7$

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In $GF(q)$, q possibilities!

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Shamir's Secret Sharing Scheme:

- Choose random deg $k-1$ poly st $p(0) = 1234$

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Shamir's Secret Sharing Scheme:

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  - Can choose points and interpolate
  - Or can choose coefficients
- Distribute $p(i)$ to ith staff member $(1 \leq i \leq n)$

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Claim: k staff members can recover password

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Proof:

- Have $k-1$ known points
- Any value of $p(0)$ gives potential polynomial
- All values consistent with known points!

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Suppose my secret is 4.
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Exercise: choose 3 pts, check Lagrange gives $x^2 + 4$

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Can modify protocol for more complicated setups

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See more examples of this in discussion

Fin

Next time: error correcting codes!