Bonus Lecture 3: Cantor-Schröder-Bernstein Theorem

Or Is It Cantor-Schröder-Berenstain?
Recall from lecture:

**Cantor-Schröder-Bernstein Theorem:**
If \( \exists \) one-to-one functions \( f : A \rightarrow B \) and \( g : B \rightarrow A \), then \( \exists \) bijection \( b : A \rightarrow B \).
Recall from lecture:

**Cantor-Schröder-Bernstein Theorem:**
If $\exists$ one-to-one functions $f : A \to B$ and $g : B \to A$, then $\exists$ bijection $b : A \to B$

How can we prove this?
Recall from lecture:

**Cantor-Schröder-Bernstein Theorem:**

If \( \exists \) one-to-one functions \( f : A \rightarrow B \) and \( g : B \rightarrow A \), then \( \exists \) bijection \( b : A \rightarrow B \)

How can we prove this?
Need to somehow combine parts of \( g \) and \( f \)
A First Attempt

Have $f : A \rightarrow B$, $g : B \rightarrow A$

Let $R_g = \{ x \in A | (\exists y \in B)(g(y) = x) \}$
A First Attempt

Have \( f : A \rightarrow B, g : B \rightarrow A \)

Let \( R_g = \{ x \in A | (\exists y \in B)(g(y) = x) \} \)

g onto \( R_g \), already known injective
Means \( g \) is bijection from \( B \rightarrow R_g \)!
A First Attempt

Have $f : A \to B$, $g : B \to A$

Let $R_g = \{ x \in A | (\exists y \in B) (g(y) = x) \}$

$g$ onto $R_g$, already known injective
Means $g$ is bijection from $B \to R_g$!

Means $g^{-1}$ is bijection from $R_g \to B$
A First Attempt

Have $f : A \rightarrow B$, $g : B \rightarrow A$

Let $R_g = \{x \in A | (\exists y \in B)(g(y) = x)\}$

g onto $R_g$, already known injective
Means $g$ is bijection from $B \rightarrow R_g$!

Means $g^{-1}$ is bijection from $R_g \rightarrow B$
Wanted bijection $A \rightarrow B$ — is close!
Quick Fix

Have map $R_g \to B$, want map $A \to B$

Where do we map $A - R_g$?
Quick Fix

Have map $R_g \rightarrow B$, want map $A \rightarrow B$

Where do we map $A - R_g$?

Only have $f$ and $g$ available, $g$ not helpful...
Quick Fix

Have map $R_g \rightarrow B$, want map $A \rightarrow B$

Where do we map $A \setminus R_g$?

Only have $f$ and $g$ available, $g$ not helpful...
So use $f$!
Quick Fix

Have map $R_g \to B$, want map $A \to B$

Where do we map $A - R_g$?

Only have $f$ and $g$ available, $g$ not helpful...
So use $f$!

First attempt: take $b(x) = \begin{cases} f(x) & x \notin R_g \\ g^{-1}(x) & x \in R_g \end{cases}$
Quick Fix

Have map $R_g \to B$, want map $A \to B$

Where do we map $A - R_g$?

Only have $f$ and $g$ available, $g$ not helpful...
So use $f$!

First attempt: take $b(x) = \begin{cases} f(x) & x \not\in R_g \\ g^{-1}(x) & x \in R_g \end{cases}$

Issue: not injective any more!
Injectivity Issues

Problem: $f$ maps $A \rightarrow R_g$ to places hit by $g^{-1}$
Injectivity Issues

Problem: $f$ maps $A \rightarrow R_g$ to places hit by $g^{-1}$

$A \rightarrow R_g$ has no where else to be mapped
So displace $x \in R_g$ that conflict!
Injectivity Issues

Problem: $f$ maps $A - R_g$ to places hit by $g^{-1}$

$A - R_g$ has no where else to be mapped
So displace $x \in R_g$ that conflict!

Formally: let $A_0 = A - R_g$, $A_1 = \{ g(f(x)) | x \in A_0 \}$

Elts in $A_1$ can’t be mapped by $g^{-1}$
Injectivity Issues

Problem: $f$ maps $A - R_g$ to places hit by $g^{-1}$

$A - R_g$ has no where else to be mapped
So displace $x \in R_g$ that conflict!

Formally: let $A_0 = A - R_g$, $A_1 = \{g(f(x)) | x \in A_0\}$

Elts in $A_1$ can’t be mapped by $g^{-1}$

Attempt 2: $b(x) = \begin{cases} f(x) & x \in (A_0 \cup A_1) \\ g^{-1}(x) & \text{ow} \end{cases}$
Injectivity Issues

Problem: \( f \) maps \( A - R_g \) to places hit by \( g^{-1} \)

\( A - R_g \) has no where else to be mapped
So displace \( x \in R_g \) that conflict!

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Elts in \( A_1 \) can’t be mapped by \( g^{-1} \)

Attempt 2: \( b(x) = \begin{cases} 
    f(x) & x \in (A_0 \cup A_1) \\
    g^{-1}(x) & ow
\end{cases} \)

Injective yet?
Injectivity Issues

Problem: \( f \) maps \( A - R_g \) to places hit by \( g^{-1} \)

\( A - R_g \) has no where else to be mapped
So displace \( x \in R_g \) that conflict!

Formally: let \( A_0 = A - R_g, A_1 = \{ g(f(x)) | x \in A_0 \} \)
Elts in \( A_1 \) can’t be mapped by \( g^{-1} \)

Attempt 2: \( b(x) = \begin{cases} f(x) & x \in (A_0 \cup A_1) \\ g^{-1}(x) & \text{otherwise} \end{cases} \)

Injective yet? Noooope!
I Have $n$ Problems

Fixed collisions w/$f(A_0)$, but now collide w/$f(A_1)$!

Collisions with $A_2 := \{g(f(x)) | x \in A_1\}$
I Have $n$ Problems

Fixed collisions w/$f(A_0)$, but now collide w/$f(A_1)$!

Collisions with $A_2 := \{g(f(x))|x \in A_1\}$

Can’t displace $A_1$ — would collide with $A_0$ again
So have to displace $A_2$
I Have $n$ Problems

Fixed collisions w/ $f(A_0)$, but now collide w/ $f(A_1)$!

Collisions with $A_2 := \{ g(f(x)) | x \in A_1 \}$

Can’t displace $A_1$ — would collide with $A_0$ again
So have to displace $A_2$

Use $f(x)$ for $x \in A_0 \cup A_1 \cup A_2$, $g^{-1}(x)$ ow
I Have n Problems

Fixed collisions w/ \( f(A_0) \), but now collide w/ \( f(A_1) \)!

Collisions with \( A_2 := \{ g(f(x)) | x \in A_1 \} \)

Can’t displace \( A_1 \) — would collide with \( A_0 \) again
So have to displace \( A_2 \)

Use \( f(x) \) for \( x \in A_0 \cup A_1 \cup A_2 \), \( g^{-1}(x) \) ow
Now collisions with \( A_2 \) and \( A_3 = \{ g(f(x)) | x \in A_2 \} \)
I Have $n$ Problems

Fixed collisions w/$f(A_0)$, but now collide w/$f(A_1)$!

Collisions with $A_2 := \{g(f(x))|x \in A_1\}$

Can’t displace $A_1$ — would collide with $A_0$ again

So have to displace $A_2$

Use $f(x)$ for $x \in A_0 \cup A_1 \cup A_2$, $g^{-1}(x)$ ow

Now collisions with $A_2$ and $A_3 = \{g(f(x))|x \in A_2\}$

...
I Have $n$ Problems

Fixed collisions w/$f(A_0)$, but now collide w/$f(A_1)$!

Collisions with $A_2 := \{g(f(x))|x \in A_1\}$

Can’t displace $A_1$ — would collide with $A_0$ again
So have to displace $A_2$

Use $f(x)$ for $x \in A_0 \cup A_1 \cup A_2$, $g^{-1}(x)$ ow
Now collisions with $A_2$ and $A_3 = \{g(f(x))|x \in A_2\}$

... 

Idea: repeat trick $ad infinitum$
Final Bijection

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{ g(f(x)) | x \in A_{i-1} \}$

Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$
Final Bijection

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{ g(f(x)) | x \in A_{i-1} \}$

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Claim: $b$ is onto $B$
Final Bijection

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For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$

Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

**Claim**: $b$ is onto $B$

- Let $y \in B$
  - Case 1: $g(y) \in A_n$ for some $n$
Final Bijection

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{ g(f(x)) \mid x \in A_{i-1} \}$

Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: $b$ is onto $B$

- Let $y \in B$

- Case 1: $g(y) \in A_n$ for some $n$
  - $n \neq 0$ since $g(y) \in R_g$
  - So $\exists x \in A_{n-1}$ s.t. $b(x) = f(x) = y$
Final Bijection

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$

Then $b(x) = \begin{cases} 
  f(x) & x \in A_n \text{ for some } n \\
  g^{-1}(x) & \text{ow}
\end{cases}$

**Claim:** $b$ is onto $B$

- Let $y \in B$
- Case 1: $g(y) \in A_n$ for some $n$
  - $n \neq 0$ since $g(y) \in R_g$
  - So $\exists x \in A_{n-1}$ s.t. $b(x) = f(x) = y$
- Case 2: $g(y) \notin A_n$ for any $n$
Final Bijection

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$

Then $b(x) = \begin{cases} 
  f(x) & x \in A_n \text{ for some } n \\
  g^{-1}(x) & \text{ow}
\end{cases}$

**Claim**: $b$ is onto $B$

- Let $y \in B$
  - Case 1: $g(y) \in A_n$ for some $n$
    - $n \neq 0$ since $g(y) \in R_g$
    - So $\exists x \in A_{n-1}$ st $b(x) = f(x) = y$
  - Case 2: $g(y) \notin A_n$ for any $n$
    - Then $b(g(y)) = g^{-1}(g(y)) = y$
Final Bijection 2

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x))| x \in A_{i-1}\}$

Then $h(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

**Claim:** $b$ is one-to-one
Final Bijection 2

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{ g(f(x)) | x \in A_{i-1} \}$

Then $h(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: $b$ is one-to-one

- Suppose have $x \neq x'$ st $b(x) = b(x')$
Final Bijection 2

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{ g(f(x)) | x \in A_{i-1} \}$

Then $h(x) = \begin{cases} 
  f(x) & x \in A_n \text{ for some } n \\
  g^{-1}(x) & \text{ow} 
\end{cases}$

Claim: $b$ is one-to-one

- Suppose have $x \neq x'$ st $b(x) = b(x')$
- $f$ injective, so can’t have $f(x) = f(x')$
- Ditto with $g^{-1}$
Final Bijection 2

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{ g(f(x)) | x \in A_{i-1} \}$

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Claim: $b$ is one-to-one

- Suppose have $x \neq x'$ st $b(x) = b(x')$
- $f$ injective, so can’t have $f(x) = f(x')$
- Ditto with $g^{-1}$
- So have $x$ in first case, $x'$ in second
Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{ g(f(x)) | x \in A_{i-1} \}$

Then $h(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

**Claim:** $b$ is one-to-one

- Suppose have $x \neq x'$ st $b(x) = b(x')$
- $f$ injective, so can’t have $f(x) = f(x')$
- Ditto with $g^{-1}$
- So have $x$ in first case, $x'$ in second
- But $f(x) = g^{-1}(x')$ means $g(f(x)) = x'$
- So $x'$ also in case 1 — contradiction!
Proof By Picture

Note: $C_i$ in diagram is our $A_i$
Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$
Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n|n \in \mathbb{N}\}$, so $A_0 = \{n|n \text{ is odd}\}$
Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n|n \in \mathbb{N}\}$, so $A_0 = \{n|n \text{ is odd}\}$
$A_1 = \{4n|n \text{ is odd}\}$
Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n|n \in \mathbb{N}\}$, so $A_0 = \{n|n \text{ is odd}\}$
$A_1 = \{4n|n \text{ is odd}\}$
$A_2 = \{16n|n \text{ is odd}\}$
Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n|n \in \mathbb{N}\}$, so $A_0 = \{n|n \text{ is odd}\}$
$A_1 = \{4n|n \text{ is odd}\}$
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...

$R_g$
Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n|n \in \mathbb{N}\}$, so $A_0 = \{n|n \text{ is odd}\}$
$A_1 = \{4n|n \text{ is odd}\}$
$A_2 = \{16n|n \text{ is odd}\}$

\[\vdots\]
$A_i = \{2^{2i}n|n \text{ is odd}\}$
Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n|n \in \mathbb{N}\}$, so $A_0 = \{n|n \text{ is odd}\}$
$A_1 = \{4n|n \text{ is odd}\}$
$A_2 = \{16n|n \text{ is odd}\}$

$\ldots$

$A_i = \{2^i n|n \text{ is odd}\}$

So $b(x) = \begin{cases} 
2x & x = 2^{2k}\text{ o st } o \text{ odd} \\
\frac{x}{2} & x = 2^{2k+1}\text{ o st } o \text{ odd} \\
0 & x = 0 
\end{cases}$
Fin

Have a great weekend!