Bonus Lecture 3: Cantor-Schröder-Bernstein Theorem

Or Is It Cantor-Schröder-Berenstain?

Recall

Recall from lecture:
Cantor-Schröder-Bernstein Theorem:
If $\exists$ one-to-one functions $f : A \to B$ and $g : B \to A$, then $\exists$ bijection $b : A \to B$

Recall

Recall from lecture:
Cantor-Schröder-Bernstein Theorem:
If $\exists$ one-to-one functions $f : A \to B$ and $g : B \to A$,
then $\exists$ bijection $b : A \to B$

How can we prove this?

Recall

Recall from lecture:
Cantor-Schröder-Bernstein Theorem:
If $\exists$ one-to-one functions $f : A \to B$ and $g : B \to A$, then $\exists$ bijection $b : A \to B$

How can we prove this?
Need to somehow combine parts of g and f

A First Attempt

Have $f\colon A \to B$, $g : B \to A$

Let $R_g = \{x \in A | (\exists y \in B)(g(y) = x)\}$

A First Attempt

Have $f : A \to B$, $g : B \to A$

Let $R_g = \{x \in A | (\exists y \in B)(g(y) = x)\}$

g onto R_g, already known injective
Means g is bijection from $B \to R_g$!

A First Attempt

Have $f\colon A \to B$, $g : B \to A$

Let $R_g = \{x \in A | (\exists y \in B)(g(y) = x)\}$

g onto R_g, already known injective
Means g is bijection from $B \to R_g$!

Means g^{-1} is bijection from $R_g \to B$

A First Attempt

Have $f\colon A \to B$, $g : B \to A$

Let $R_g = \{x \in A | (\exists y \in B)(g(y) = x)\}$

g onto R_g, already known injective
Means g is bijection from $B \to R_g$!

Means g^{-1} is bijection from $R_g \to B$
Wanted bijection $A \to B$ — is close!

Quick Fix

Have map $R_g \to B$, want map $A \to B$

Where do we map $A - R_g$?

Quick Fix

Have map $R_g \to B$, want map $A \to B$

Where do we map $A - R_g$?

Only have f and g available, g not helpful...

Quick Fix

Have map $R_g \to B$, want map $A \to B$

Where do we map $A - R_g$?

Only have f and g available, g not helpful...
So use f!

Quick Fix

Have map $R_g \to B$, want map $A \to B$

Where do we map $A - R_g$?

Only have f and g available, g not helpful...
So use f!

First attempt: take $b(x) = \begin{cases} f(x) & x \notin R_g \\ g^{-1}(x) & x \in R_g \end{cases}$

Quick Fix

Have map $R_g \to B$, want map $A \to B$

Where do we map $A - R_g$?

Only have f and g available, g not helpful...
So use f!

First attempt: take $b(x) = \begin{cases} f(x) & x \notin R_g \\ g^{-1}(x) & x \in R_g \end{cases}$

Issue: not injective any more!

Injectivity Issues

Problem: f maps $A - R_g$ to places hit by g^{-1}

Injectivity Issues

Problem: f maps $A - R_g$ to places hit by g^{-1}

$A - R_g$ has no where else to be mapped
So displace $x \in R_g$ that conflict!

Injectivity Issues

Problem: f maps $A - R_g$ to places hit by g^{-1}

$A - R_g$ has no where else to be mapped
So displace $x \in R_g$ that conflict!

Formally: let $A_0 = A - R_g$, $A_1 = \{g(f(x)) | x \in A_0\}$

Elts in A_1 can't be mapped by g^{-1}

Injectivity Issues

Problem: f maps $A - R_g$ to places hit by g^{-1}

$A - R_g$ has no where else to be mapped
So displace $x \in R_g$ that conflict!

Formally: let $A_0 = A - R_g$, $A_1 = \{g(f(x))|x \in A_0\}$

Elts in A_1 can't be mapped by g^{-1}

Attempt 2: $b(x) = \begin{cases} f(x) & x \in (A_0 \cup A_1) \\ g^{-1}(x) & \text{ow} \end{cases}$

Injectivity Issues

Problem: f maps $A - R_g$ to places hit by g^{-1}

$A - R_g$ has no where else to be mapped
So displace $x \in R_g$ that conflict!

Formally: let $A_0 = A - R_g$, $A_1 = \{g(f(x)) | x \in A_0\}$

Elts in A_1 can't be mapped by g^{-1}

Attempt 2: $b(x) = \begin{cases} f(x) & x \in (A_0 \cup A_1) \\ g^{-1}(x) & \text{ow} \end{cases}$

Injective yet?

Injectivity Issues

Problem: f maps $A - R_g$ to places hit by g^{-1}

$A - R_g$ has no where else to be mapped
So displace $x \in R_g$ that conflict!

Formally: let $A_0 = A - R_g$, $A_1 = \{g(f(x)) | x \in A_0\}$

Elts in A_1 can't be mapped by g^{-1}

Attempt 2: $b(x) = \begin{cases} f(x) & x \in (A_0 \cup A_1) \\ g^{-1}(x) & \text{ow} \end{cases}$

Injective yet? Nope!

I Have n Problems

Fixed collisions w/ $f(A_0)$, but now collide w/ $f(A_1)$!

Collisions with $A_2 := \{g(f(x)) | x \in A_1\}$

I Have n Problems

Fixed collisions w/$f(A_0)$, but now collide w/$f(A_1)$!

Collisions with $A_2 := \{g(f(x)) | x \in A_1\}$

Can't displace A_1 — would collide with A_0 again
So have to display A_2

I Have n Problems

Fixed collisions w/$f(A_0)$, but now collide w/$f(A_1)$!

Collisions with $A_2 := \{g(f(x))|x \in A_1\}$

Can't displace A_1 — would collide with A_0 again
So have to displace A_2

Use $f(x)$ for $x \in A_0 \cup A_1 \cup A_2$, $g^{-1}(x)$ ow

I Have n Problems

Fixed collisions w/ $f(A_0)$, but now collide w/ $f(A_1)$!

Collisions with $A_2 := \{g(f(x)) | x \in A_1\}$

Can't displace A_1 — would collide with A_0 again
So have to displace A_2

Use $f(x)$ for $x \in A_0 \cup A_1 \cup A_2$, $g^{-1}(x)$ ow
Now collisions with A_2 and $A_3 = \{g(f(x)) | x \in A_2\}$

I Have n Problems

Fixed collisions w/$f(A_0)$, but now collide w/$f(A_1)$!

Collisions with $A_2 := \{g(f(x))|x \in A_1\}$

Can't displace A_1 — would collide with A_0 again
So have to displace A_2

Use $f(x)$ for $x \in A_0 \cup A_1 \cup A_2$, $g^{-1}(x)$ ow
Now collisions with A_2 and $A_3 = \{g(f(x))|x \in A_2\}$
...

I Have *n* Problems

Fixed collisions w/$f(A_0)$, but now collide w/$f(A_1)$!

Collisions with $A_2 := \{g(f(x)) | x \in A_1\}$

Can't displace A_1 — would collide with A_0 again
So have to displace A_2

Use $f(x)$ for $x \in A_0 \cup A_1 \cup A_2$, $g^{-1}(x)$ ow
Now collisions with A_2 and $A_3 = \{g(f(x)) | x \in A_2\}$
...

Idea: repeat trick *ad infinitum*

Final Bijection

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$

Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Final Bijection

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$
Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is onto B

Final Bijection

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$
Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is onto B

- Let $y \in B$
- Case 1: $g(y) \in A_n$ for some n

Final Bijection

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$
Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is onto B

- Let $y \in B$
- Case 1: $g(y) \in A_n$ for some n
  - $n \neq 0$ since $g(y) \in R_g$
  - So $\exists x \in A_{n-1}$ st $b(x) = f(x) = y$

Final Bijection

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$
Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is onto B

- Let $y \in B$
- Case 1: $g(y) \in A_n$ for some n
  - $n \neq 0$ since $g(y) \in R_g$
  - So $\exists x \in A_{n-1}$ st $b(x) = f(x) = y$
- Case 2: $g(y) \notin A_n$ for any n

Final Bijection

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$

Then $b(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is onto B

- Let $y \in B$
- Case 1: $g(y) \in A_n$ for some n
  - $n \neq 0$ since $g(y) \in R_g$
  - So $\exists x \in A_{n-1}$ st $b(x) = f(x) = y$
- Case 2: $g(y) \notin A_n$ for any n
  - Then $b(g(y)) = g^{-1}(g(y)) = y$

Final Bijection 2

Let $A_0 = A - R_g$

For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$

Then $h(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is one-to-one

Final Bijection 2

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$
Then $h(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is one-to-one

- Suppose have $x \neq x'$ st $b(x) = b(x')$

Final Bijection 2

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$
Then $h(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is one-to-one

- Suppose have $x \neq x'$ st $b(x) = b(x')$
- f injective, so can't have $f(x) = f(x')$
- Ditto with g^{-1}

Final Bijection 2

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$
Then $h(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is one-to-one

- Suppose have $x \neq x'$ st $b(x) = b(x')$
- f injective, so can't have $f(x) = f(x')$
- Ditto with g^{-1}
- So have x in first case, x' in second

Final Bijection 2

Let $A_0 = A - R_g$
For $i \geq 1$, let $A_i = \{g(f(x)) | x \in A_{i-1}\}$

Then $h(x) = \begin{cases} f(x) & x \in A_n \text{ for some } n \\ g^{-1}(x) & \text{ow} \end{cases}$

Claim: b is one-to-one

- Suppose have $x \neq x'$ st $b(x) = b(x')$
- f injective, so can't have $f(x) = f(x')$
- Ditto with g^{-1}
- So have x in first case, x' in second
- But $f(x) = g^{-1}(x')$ means $g(f(x)) = x'$
- So x' also in case 1 — contradiction!

Proof By Picture

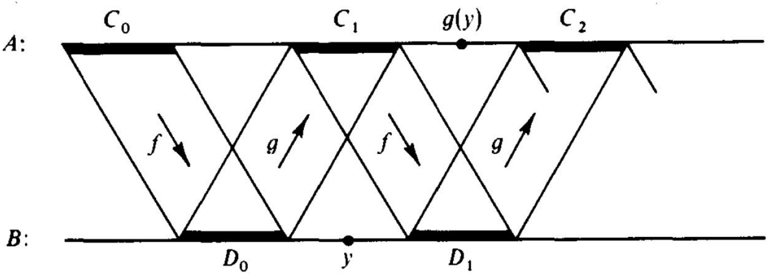

Photo Credit: *Elements of Set Theory* by Herbert Enderton

Note: C_i in diagram is our A_i

Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n | n \in \mathbb{N}\}$, so $A_0 = \{n | n \text{ is odd}\}$

Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n | n \in \mathbb{N}\}$, so $A_0 = \{n | n \text{ is odd}\}$
$A_1 = \{4n | n \text{ is odd}\}$

Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n | n \in \mathbb{N}\}$, so $A_0 = \{n | n \text{ is odd}\}$
$A_1 = \{4n | n \text{ is odd}\}$
$A_2 = \{16n | n \text{ is odd}\}$

Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n | n \in \mathbb{N}\}$, so $A_0 = \{n | n \text{ is odd}\}$
$A_1 = \{4n | n \text{ is odd}\}$
$A_2 = \{16n | n \text{ is odd}\}$

...

Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n | n \in \mathbb{N}\}$, so $A_0 = \{n | n \text{ is odd}\}$
$A_1 = \{4n | n \text{ is odd}\}$
$A_2 = \{16n | n \text{ is odd}\}$
...
$A_i = \{2^{2^i} n | n \text{ is odd}\}$

Example Application

Example of CSB in action:
Take $A = B = \mathbb{N}$, $f(x) = g(x) = 2x$

$R_g = \{2n | n \in \mathbb{N}\}$, so $A_0 = \{n | n \text{ is odd}\}$
$A_1 = \{4n | n \text{ is odd}\}$
$A_2 = \{16n | n \text{ is odd}\}$
...
$A_i = \{2^{2i} n | n \text{ is odd}\}$

$$\text{So } b(x) = \begin{cases} 2x & x = 2^{2k} o \text{ st } o \text{ odd} \\ \frac{x}{2} & x = 2^{2k+1} o \text{ st } o \text{ odd} \\ 0 & x = 0 \end{cases}$$

Fin

Have a great weekend!