The Catalan Numbers

CS 70, Summer 2019

Bonus Lecture, 7/19/19

Parentheses

How many ways can I **properly** arrange:

- Zero pairs of parentheses?
- One pair of parentheses?
- Two pairs of parentheses?
- Three pairs of parentheses?
- Four pairs of parentheses? Already getting hard...

Five Pairs?

Catalan Number Formula

The n-th Catalan number is given by:

$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n}$$

We'll see how to get this formula later...

Why is C_n **always an integer**? We can rewrite:

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}$$

Recursion In Parentheses

The first character will always be a **left parenthesis.**

1. Identify the **right parenthesis** it is matched with.
2. What goes **inside** the first left parenthesis and its partner? How many of them?
3. What goes **after** its partner? How many of them?

Recursion In Parentheses: Count By Cases

Cases based on **how many pairs of parentheses inside**:

- Case 0: No pairs inside.
- Case 1: One pair inside.
- Case 2: Two pairs inside.
- Case i: i pairs inside.

Catalan Number Recurrence

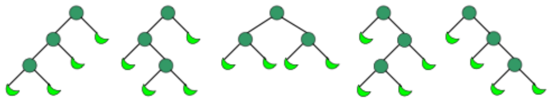
The n-th Catalan number is also given by:

$$C_n = \sum_{i=0}^{n-1} C_i \cdot C_{(n-1)-i}$$

If an object satisfies this recurrence, it can be counted by C_n!

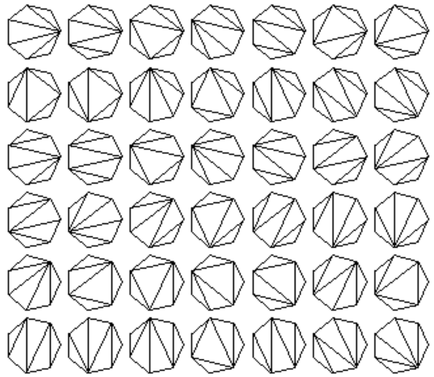
Example: **Full** binary trees with $(n+1)$ leaves

Def: Every vertex either has 2 children, or no children



Other Items Counted By Recurrence

Example: **Triangulations** in $(n+2)$-sided polygons



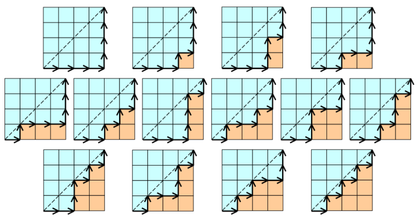
Counting Using Bijections

Example: **Bitstrings** of n 0's and n 1's where, if we read from left to right, there are always more 0's than 1's.

Q: Bijection to something that we already counted?

Counting Using Bijections

Example: **Lattice paths** on an $n \times n$ grid which do not cross above the diagonal.



Q: Bijection to something that we already counted?

Deriving the Formula

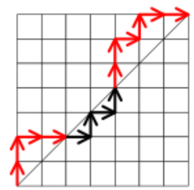
Let's study the formula again...

$$C_n = \frac{1}{n+1} \cdot \binom{2n}{n}$$

- What kinds of objects does $\binom{2n}{n}$ count?
- What does the $\frac{1}{n+1}$ clue into?

Exceedence

Exceedence: the number of **vertical edges** in the lattice path that lie **above the diagonal**



Exceedence $= 0$:

How many possible values are there for exceedence?

Bijection Between Exceedences

Goal: We can show that 0-exceedence paths are counted by C_n if:

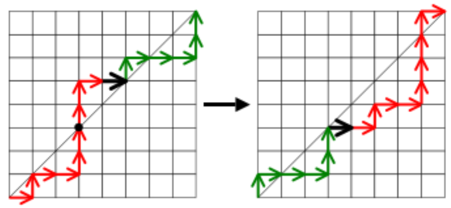
$$\#\{0\text{-exc.}\} = \#\{1\text{-exc.}\} = \#\{2\text{-exc.}\} = \ldots = \#\{n\text{-exc.}\}$$

Want a bijection (**read: invertible transformation**) from

$$\#\{(i+1)\text{-exc.}\} \text{ to } \{i\text{-exc.}\}$$

Bijection Between Exceedences

- Follow path **under diagonal** until it first goes above the diagonal. (It can be in the corner!)
- Mark the intersection with the diagonal.
- Continue following (now **above the diagonal**) until we hit the diagonal again. Mark edge *e* that occurs before this hit
- Swap the portion before *e* and the portion after *e*.



Summary

- All about the **recursive structure**
  - Complicated objects actually have **nice structure**
  - Reduces problem to a **smaller version of itself**
- Can also define **bijections** between interesting objects
  - Lets you count a huge variety of objects